CLASSICAL AND QUANTUM TWO-BODY PROBLEM IN GENERAL RELATIVITY*

A. MAHESHWARI**
International Centre for Theoretical Physics, Trieste, Italy

E.R. NISSIMOV*** and I.T. TODOROV***
International Centre for Theoretical Physics, Trieste, Italy and Scuola Internazionale Superiore di Studi Avanzati, Trieste, Italy

ABSTRACT. The two-body problem in general relativity is reduced to the problem of an effective particle (with an energy-dependent relativistic reduced mass) in an external field. The effective potential is evaluated from the Born diagram of the linearized quantum theory of gravity. It reduces to a Schwarzschild-like potential with two different 'Schwarzschild radii'. The results derived in a weak field approximation are expected to be relevant for relativistic velocities.

1. In both non-relativistic and special relativistic mechanics, classical and quantum, the two-body problem for (spinless) point particles is reduced to the conceptually simpler problem of a single effective particle moving in an external field. The only exception to this picture so far seems to be the general theory of relativity, where the two-body problem has been treated in a considerably more complicated way: as a field-theoretic problem with singularities [1, 2] (or as a problem of finite size bodies interacting with a gravitational field [3]).

Here we propose to treat gravitational two-particle interaction in much the same way as electromagnetic interactions have been tackled previously [4, 5] in the quasipotential approach [6] which found its natural place in the constraint Hamiltonian framework of References [7] and [8].**** Unlike other first-order (in $1/c^2$) semi-relativistic treatments (based on a quantum field theoretic derivation of the two-particle potential) [12], our approach is fully relativistic. Here we shall consider the two-body problem in the leading order of perturbation theory in G, the Newtonian gravitational constant. It is reduced to the problem of an effective particle (with an energy-dependent relativistic reduced mass) in an external Schwarzschild-like field with two different 'Schwarzschild radii', in g_{00} and g_{ij}, respectively.

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**Permanent address: Department of Physics, Regional College of Education, Mysore-6, 570006, India.

***Permanent address: Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Lenin 72, 1184 Sofia, Bulgaria.

****A similar approach is being developed by a number of authors (see, e.g., References [9] and [10]); the reader will find a comprehensive bibliography in References [11] and [8].

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2. We shall briefly summarize the constraint Hamiltonian approach to the relativistic two-body problem and will introduce the notion of an effective particle in this approach.

We define the generalized two-point (spinless) particle mass shell as a 14-dimensional submanifold of the 16-dimensional 'large phase space' Γ of Minkowski space co-ordinates x_1, x_2 and four-momenta p_1, p_2, given by two first-class constraints. We postulate (as in [4, 5]) that the difference $p_1^2 - p_2^2$ is independent of the interaction:

$$\varphi = \tfrac{1}{2}(m_1^2 + p_1^2 - m_2^2 - p_2^2) = pP = 0, \tag{1}$$

where m_1, m_2 are the masses of the two particles, P and p are the total and the relative momenta:

$$P = p_1 + p_2, \qquad p = \mu_1 p_2, \qquad \mu_1 + \mu_2 = 1,$$
$$\mu_1 - \mu_2 = \frac{m_1^2 - m_2^2}{w^2}, \quad w^2 = -P^2 (> 0). \tag{2}$$

(We are using the space-like signature $-+++$ for the metric tensor.)

The non-relativistic reduced mass m is defined by the equation $mM = m_1 m_2$, where $M = m_1 + m_2$ is the total mass. We use the same equation to define *the relativistic reduced mass* m_w, just replacing M by the *total relativistic mass* w $(= (-P^2)^{1/2})$:

$$m_w = \frac{m_1 m_2}{w}. \tag{3}$$

The effective particle four-momentum P_{eff} is then defined in the centre-of-mass frame (in which $P = (w, \mathbf{0}), p = (0, \mathbf{p})$) by

$$p_{\text{eff}} = (E, \mathbf{p}), \qquad E = (m_w^2 + b^2(w))^{1/2} = \frac{w^2 - m_1^2 - m_2^2}{2w}, \tag{4}$$

where $b^2(w)$ is the one-shell value of the relative momentum square

$$b^2(w) = \frac{w^4 - 2(m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2}{4w^2}. \tag{5}$$

In the first approximation in the coupling constants (charges) $e_{1,2}$, the electromagnetic interaction of two charged particles has been given by the Hamiltonian constraint [4, 5, 8]

$$H_{\text{Coul}} = \tfrac{1}{2}[m_w^2 + \mathbf{p}^2 - (E - V_{\text{Coul}})^2] = 0, \quad \mathbf{p}^2 = p^2 = p_{\text{eff}}^2 + \frac{(Pp_{\text{eff}})^2}{w^2},$$

$$V_{\text{Coul}} = \frac{e_1 e_2}{4\pi r}, \quad r = (x_\perp^2)^{1/2} = \left(x^2 + \frac{(xP)^2}{w^2}\right)^{1/2}, \quad x = x_1 - x_2. \tag{6}$$

(Note that the constraint (6) is manifestly a Poincaré invariant; no semi-relativistic approximation of the type of the $1/c^2$ expansion has been made.) The idea of the present note is to describe in a similar fashion the gravitational interaction of two relativistic masses by setting

$$H = H_{\text{Grav}} = \tfrac{1}{2}\,[m_w^2 + g^{\mu\nu} P_{\text{eff}\mu} P_{\text{eff}\nu}] = 0, \tag{7}$$

where $g^{\mu\nu}$ is some appropriate modification of the Schwarzschild metric.

3. The actual computation of the electromagnetic Hamiltonian constraint (which includes corrections to H_{coul}) has been effected in the quasipotential approach to quantum electrodynamics [4, 5]. We shall pursue here a similar path starting with a standard linearized form of quantum gravity (cf. References [13]).

According to Dirac's general theory [14], the quantum counterpart of the first-class constraint (7) is the relativistic 'Schrödinger equation'

$$[m_w^2 + \tfrac{1}{6}R - |g|^{-1/2}\partial_\mu(|g|^{1/2} g^{\mu\nu})\partial_\nu]\,\Psi = 0 \tag{8}$$

for the state vector $\Psi(x)$. Here R is the scalar curvature. (As pointed out by Penrose [15], the $R/6$ term is necessary in order to ensure conformal invariance of the zero mass limit.) [The Laplace–Beltrami operator provides the appropriate generally covariant ordering of the canonical variables x^μ_{eff} $(= -(Px)w^{-2}P^\mu + x^\mu_\perp)$ and $p_{\text{eff}\mu} = -i\partial_\mu$.] The momentum space counterpart of (8) is to be identified with the local quasipotential equation [4, 5] (written here in the centre of the mass frame)

$$\begin{aligned} &G_w^{-1}(\mathbf{p})\widetilde{\Psi}(\mathbf{p}) + (V_*\widetilde{\Psi})(\mathbf{p}) \\ &\equiv 2w[\mathbf{p}^2 - b^2(w)]\,\widetilde{\Psi}(\mathbf{p}) + \int V(\mathbf{p},\mathbf{q})\widetilde{\Psi}(\mathbf{q})\,\frac{d^3q}{(2\pi)^3} = 0; \end{aligned} \tag{9}$$

the potential V is determined order by order in G from the Lippmann–Schwinger-type equation:

$$T + V + V_* G_w T = 0, \qquad G_w(\mathbf{k}) = [2w(\mathbf{k}^2 - b^2(w) - iO)]^{-1} \tag{10}$$

and from the Feynman expansion of the scattering amplitude $T = T_w(\mathbf{p}, \mathbf{q})$ in a quantum theory of gravitationally-interacting scalar particles.

We shall treat Equations (8) and (9) in the leading order approximation of perturbation theory. The linearized form of (8) is obtained by setting

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \text{diag}\,(-1, +1, +1, +1), \qquad |h_{\mu\nu}| \ll 1, \tag{11}$$

and using $g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}$, $|g| \approx 1 + h^\mu_\mu$, $R \approx (\partial_\mu\partial_\nu h^{\mu\nu} - \Box h^\mu_\mu)$ (where Lorentz indices are raised and lowered by η). Up to terms of order $O(h^2)$ Equation (8) reads:

$$\{m_w^2 - \Box + [h^{\mu\nu}\partial_\mu\partial_\nu + \tfrac{1}{6}(\partial_\mu\partial_\nu h^{\mu\nu}) + (\partial_\mu h^{\mu\nu} - \tfrac{1}{2}\partial^\nu h^\lambda_\lambda)\partial_\nu - \tfrac{1}{6}(\Box h^\mu_\mu]\}\Psi = 0. \tag{12}$$

Thus, in the leading order of perturbation theory we have:

$$\begin{aligned}&\widetilde{h}^{\mu\nu}(p-q)q_\mu q_\nu + \tfrac{1}{6}(p-q)_\mu(p-q)_\nu \widetilde{h}^{\mu\nu}(p-q) + \\ &+ [(p-q)_\mu \widetilde{h}^{\mu\nu}(p-q) - \tfrac{1}{2}(p-q)^\nu \widetilde{h}^\lambda_\lambda(p-q)] \times \\ &\times q_\nu - \tfrac{1}{6}(p-q)^2 \widetilde{h}^\mu_\mu(p-q) = \frac{1}{2w} T^{(1)}_w(\mathbf{p}, \mathbf{q}),\end{aligned} \tag{13}$$

$$p = (p_\mu) = (-E, \mathbf{p}), q = (-E, \mathbf{q}),$$

where E is given by (4) (and $\mathbf{p}^2 = \mathbf{q}^2 = b^2(w)$ (on the mass shell).

4. The Born approximation $T^{(1)}_w$ for the two-particle scattering amplitude is derived from the Lagrangian density

$$\mathcal{L} = -|g|^{1/2}\left[\frac{R}{16\pi G} + \frac{1}{2}\sum_{k=1,2}(g^{\mu\nu}\partial_\mu\Phi_k\partial_\nu\Phi_k + m_k^2\Phi_k^2 + \tfrac{1}{6}R\Phi_k^2)\right] \tag{14}$$

in the weak field approximation (11).* The expression (14) differs by the $\frac{1}{6}R\Phi_k^2$ term from the Lagrangian used in References [13] (corresponding to the $R/6$ in the Schrödinger Equation (8)). The one-graviton exchange diagram between particles 1 and 2 gives

$$T^{(1)}_w(\mathbf{p}, \mathbf{q}) = 4\pi G \Gamma^{(1)}_{\kappa\lambda} \widetilde{\mathscr{D}}^{\kappa\lambda,\mu\nu}(\mathbf{p}^{(1)} - q^{(1)})\Gamma^{(2)}_{\mu\nu}, \tag{15}$$

$$\begin{aligned}\Gamma^{(k)}_{\mu\nu} &= i[p^{(k)}_\mu q^{(k)}_\nu + p^{(k)}_\nu q^{(k)}_\mu - \eta_{\mu\nu}(p^{(k)}q^{(k)} + \tfrac{1}{3}(p^{(k)} - q^{(k)})^2 + m_k^2) + \\ &+ \tfrac{1}{3}(p^{(k)} - q^{(k)})_\mu (p^{(k)} - q^{(k)})_\nu], \quad k = 1, 2,\end{aligned} \tag{16}$$

$$\widetilde{\mathscr{D}}^{\kappa\lambda,\mu\nu}_{(k)} = \frac{\eta^{\kappa\mu}\eta^{\lambda\nu} + \eta^{\kappa\nu}\eta^{\lambda\mu} - \eta^{\kappa\lambda}\eta^{\mu\nu}}{k^2 - iO}, \tag{17}$$

$$p^{(1)} = (E_1, \mathbf{p}), \quad p^{(2)} = (E_2, -\mathbf{p}), \quad q^{(1)} = (E_1, \mathbf{q}), \quad q^{(2)} = (E_2, -\mathbf{q}), \quad E_k = \mu_k w. \tag{18}$$

Inserting (16), (17) and (18) into Equation (15), we obtain:

$$T^{(1)}_w(\mathbf{p}, \mathbf{q}) = 16\pi G\left[\frac{2E^2w^2 - m_1^2 m_2^2}{(\mathbf{p} - \mathbf{q})^2} - Ew - \frac{m_1^2 + m_2^2}{6} + \frac{1}{12}(\mathbf{p} - \mathbf{q})^2\right]. \tag{19}$$

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*The naive G-perturbation theory of (14) is nonrenormalizable. According to the general discussion in Reference [16], Equation (14) gives a correct description of gravitational interactions *only* on tree-graph level and at a relatively low energy scale (much less that 10^{19} GeV for elementary particles). In order to compute V consistently to arbitrary orders in G from Equation (10) one should use a nontrivial *renormalizable* extension of (14) if there is any (at present only extended supergravity is a hopeful candidate).

5. The next step is to evaluate $h_{\mu\nu}$ from Equations (13) and (19). To this end we shall use the Euclidean invariant 'stationary gauge' in which

$$h_{0i} = 0, \qquad h_{00} = \frac{r_t}{r} \quad (r_t = \text{const}), \qquad h_{ij} = B(r) x_i x_j \quad (\text{for } x \neq 0). \tag{20}$$

(The last condition means that we require the angular part of ds^2 to have its flat space form $r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$, which is the standard co-ordinate choice for the Schwarzschild solution.) This amounts to setting

$$\begin{aligned} h_{00}(\mathbf{p}-\mathbf{q}) &= \frac{4\pi r_t}{(\mathbf{p}-\mathbf{q})^2}, \\ h_{ij}(\mathbf{p}-\mathbf{q}) &= 4\pi r_s \frac{(\mathbf{p}-\mathbf{q})^2 \delta_{ij} - 2(p_i - q_i)(p_j - q_j)}{(\mathbf{p}-\mathbf{q})^4} + C\delta_{ij}, \end{aligned} \tag{21}$$

where r_t, r_s and C are constants of the motion. Inserting (19) and (21) into Equation (13), we find*

$$r_t = 2Gw\left[1 - \frac{4b^2}{m_w^2}\left(2\frac{E}{W} - 3\frac{b^2}{w^2}\right)\right], \tag{22a}$$

$$r_s = 2Gw\left[1 + 4\frac{E^2}{m_w^2}\left(2\frac{E}{W} - 3\frac{b^2}{w^2}\right)\right]; \qquad C = -\frac{8\pi G}{w}. \tag{22b}$$

Thus, we end up with the following x-space expression for the metric tensor:

$$g_{00} = -\left(1 - \frac{r_t}{r}\right), \qquad g_{0i} = 0, \qquad g_{ij} = \delta_{ij} + r_s \frac{x_i x_j}{r^3} - \frac{8\pi G}{w}\delta_{ij}\delta(\mathbf{x}). \tag{23}$$

The last (δ-function) term does not contribute to the classical motion and will be ignored in the sequel (it may only be relevant for a quantum s-wave effect). Clearly, in the text body limit, i.e., for $(m_1 + m_2)^2 \gg m_1 m_2$, and for $|w(m_1 + m_2)^{-1} - 1| \ll 1$ (slow motion), the right-hand sides of Equations (23) go into the linearized Schwarzschild solution (both r_t and r_s tending to the Schwarzschild radius $2(m_1 + m_2)G$).

6. We are now prepared to treat the classical gravitational two-body problem by inserting the metric (23) into the Hamiltonian constraint (7). Going to spherical co-ordinates, we can rewrite Equation (7) in the form

$$H = \tfrac{1}{2}\left[m_w^2 - \left(1 - \frac{r_t}{r}\right)^{-1} p_0^2 + \left(1 - \frac{r_s}{r}\, p_r^2 + \frac{1}{r^2}\, p_\theta^2 + \frac{1}{\sin^2\theta} p_\varphi^2\right)\right] \approx 0. \tag{24}$$

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*The expression for r_s does not coincide with the correct semi-relativistic approximation of Reference [1]. The results of Reference [12] indicate that the agreement will be restored if one takes into account the semi-relativistic contribution to the effective potential coming from the Feynman diagrams of order G^2.

A standard computation using the initial condition $\theta = \pi/2, p_\theta = 0$ (cf. [17]) gives:

$$-p_0 = E\ (\dot{E} = 0), \qquad p_\varphi = J\ (\dot{J} = 0), \qquad p_r^2\left(1 - \frac{r_t}{r}\right) = b^2 + E^2 \frac{r_t}{r}\left(1 - \frac{r_t}{r}\right)^{-1} - \frac{J^2}{r^2}. \tag{25}$$

Introducing the radius variable $u = r^{-1}$ and setting $\mathrm{d}u/\mathrm{d}\varphi = u'$ we obtain

$$J^2(u'^2 + u^2 - r_s u^3) + [r_s b^2 - r_t E^2 (1 - r_s u)(1 - r_t u)^{-1}]u = b^2. \tag{26}$$

We look for a solution of this equation of the form

$$u = \ell^{-1}[1 + \epsilon(\cos \eta\varphi + f(\eta\varphi))], \tag{27}$$

where the natural dimensionless small parameter is now r_s/ℓ. The unknown function $f(\varphi)$ is expected to be a small correction (of order r_s/ℓ) to the Schwarzschild-like solution. Inserting in (26) and comparing the coefficients of $\cos 2\eta\varphi$, $\cos \eta\varphi$, and the constant term, we find:

$$\eta = 1 - \frac{3r_t}{2\ell} + \frac{r_t - r_s}{2\ell}, \qquad \epsilon^2 = 1 + \frac{4r_t}{\ell} + \frac{\ell^2 b^2}{J^2}\left(1 + \frac{3r_t + r_s}{\ell}\right),$$
$$\ell = 2J^2(r_t m_w^2 - r_s b^2)^{-1} + O(r_s). \tag{28}$$

The terms containing f, f' and $\cos^3 \eta\varphi$ lead to the differential equation

$$\sin \eta\varphi \cdot f' - \cos \eta\varphi \cdot f + \frac{\epsilon r_s}{2\ell} \cos^3 \eta\varphi = 0. \tag{29}$$

Its solution, satisfying $f < |\cos \eta\varphi|$ for all φ is

$$f = \frac{\epsilon r_s}{2\ell}(1 - |\sin \eta\varphi|)^2 \left(+ 0\left[\left(\frac{r_s}{\ell}\right)^2\right]\right) \tag{30}$$

(which is of order r_s/ℓ in accord with our expectation).

The solution (27), (28) and (30) so obtained, reduces to the classical one [1] in the semi-relativistic and test body limit.* It is expected to give a more accurate description of the two-particle motion for relativistic velocities and weak gravitational forces.

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* This is to be contrasted with the results of some previous first-order (in G) relativistic approaches [18] which give incorrect values for η even in the test body limit.

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